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LETTER TO THE EDITOR

Moment Problems and Stability Results for Frames with Applications to Irregular Sampling and Gabor Frames

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Abstract—We show the existence of a "best approximation solution" to the set of equations $\langle f, f_i\rangle = a_i, i \in \mathbf{I}$, where $\{f_i\}_{i\in\mathbf{I}}$ is a frame for a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and $\{a_i\}_{i\in\mathbf{I}} \in l^2(\mathbf{I})$. We derive formulas showing how the solution changes if $\{a_i\}_{i\in\mathbf{I}}$ or $\{f_i\}_{i\in\mathbf{I}}$ is perturbed. We explain why the results are important for irregular sampling and show how to obtain concrete estimates in this case. We also give an application to Gabor frames.

1. INTRODUCTION

Given a square summable sequence $\{a\}_{i\in\mathbf{I}}$ and a frame $\{f_i\}_{i\in\mathbf{I}}$ for a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ we consider a set of equations of the form

$$\langle f, f_i\rangle = a_i, \quad \forall i \in \mathbf{I}. \tag{1}$$

Although (1) does not need to have a solution we show in Section 2 that there always exists a "best approximation solution" in a sense to be made precise.

Using an explicit formula for this best approximation solution we are able to measure how it changes if $\{a_i\}_{i\in\mathbf{I}}$ or $\{f_i\}_{i\in\mathbf{I}}$ is perturbed. In the case where $\{f_i\}_{i\in\mathbf{I}}$ is perturbed the norm of a certain "perturbation operator K" plays the central role in our result. K also appears in a perturbation theorem giving conditions implying that a family $\{g_i\}_{i\in\mathbf{I}}$ "close" to a frame is a frame itself.

These results are important for irregular sampling, so in Section 3 we discuss frames consisting of translated versions $\{\mathrm{sinc}(\cdot - x_i)\}$ of the sinc function; here $\{x_i\}$ is the sampling set. In practice, $\{x_i\}$ can be some measured data, e.g., of the time. But due to measurement errors, $\{x_i\}$ are not always exact. However, we show that if the measured points $\{x_i\}$ and the exact points $\{y_i\}$ are sufficiently close, then $\{\mathrm{sinc}(\cdot - y_i)\}$ builds a frame if $\{\mathrm{sinc}(\cdot - x_i)\}$ does. And we derive an estimate of the error which appear if we solve a moment problem using the frame $\{\mathrm{sinc}(\cdot - x_i)\}$ instead of the frame $\{\mathrm{sinc}(\cdot - y_i)\}$. In Section 4 we apply the results from Section 2 to Gabor frames.

In the rest of Section 1 we collect some definitions and results needed later.

Let $\mathcal{H} \neq \{0\}$ be a separable Hilbert space, with the inner product $\langle\cdot,\cdot\rangle$ linear in the first entry. $\mathbf{I}$ will denote a countable index set.

A family $\{f_i\}_{i\in\mathbf{I}}$ of elements in $\mathcal{H}$ is called a *Bessel sequence* if

$$\exists B > 0 : \sum_{i\in\mathbf{I}} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2}$$

Given a Bessel sequence $\{f_i\}_{i\in\mathbf{I}}$ one can define a bounded linear operator

$$T : l^2(\mathbf{I}) \to \mathcal{H}, \quad T\{c_i\} := \sum_{i\in\mathbf{I}} c_i f_i. \tag{3}$$

Then $\|T\| \leqslant \sqrt{B}$. The adjoint operator is

$$T^* : \mathcal{H} \to l^2(\mathbf{I}), \quad T^*f = \{\langle f, f_i\rangle\}_{i\in\mathbf{I}}.$$

The *frame operator* is defined by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf := TT^*f = \sum_{i\in\mathbf{I}} \langle f, f_i\rangle f_i.$$

A Bessel sequence $\{f_i\}_{i\in\mathbf{I}}$ is called a *frame* if

$$\exists A > 0 : A\|f\|^2 \leqslant \sum_{i\in\mathbf{I}} |\langle f, f_i\rangle|^2, \quad \forall f \in \mathcal{H}. \tag{4}$$

Any pair of numbers A and B such that (2) and (4) are satisfied will be called a *set of frame bounds.* If $\{f_i\}_{i\in\mathbf{I}}$ is a frame, then S has a bounded inverse, defined on all of $\mathcal{H}$; this fact leads to the important *frame decomposition*

$$f = SS^{-1}f = \sum_{i\in\mathbf{I}} \langle S^{-1}f, f_i\rangle f_i = \sum_{i\in\mathbf{I}} \langle f, S^{-1}f_i\rangle f_i, \quad \forall f \in \mathcal{H}.$$

$\{S^{-1}f_i\}_{i\in\mathbf{I}}$ is also a frame, usually called the *dual frame;* as bounds one can use $1/B$ and $1/A$.

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2. MOMENT PROBLEMS FOR FRAMES

Let $\{f_i\}_{i\in\mathbf{I}}$ be a family of elements in $\mathcal{H}$ and $\{a_i\}_{i\in\mathbf{I}} \in l^2(\mathbf{I})$. We ask whether we can find $f \in \mathcal{H}$ such that

$$\langle f, f_i\rangle = a_i, \quad \forall i \in \mathbf{I}. \tag{1}$$

A problem of this type is called a *moment problem.* It is clear that a moment problem does not need to have a solution; in [9, Theorem 4.2] it is shown that if $\mathbf{I} = \mathbf{N}$ then (1) has a solution with $\|f\| \leqslant c$ if and only if $|\sum_{i=1}^n c_i\overline{a_i}| \leqslant c \cdot \|\sum_{i=1}^n c_i f_i\|$ for every sequence $c_1, c_2, \ldots, c_n (n = 1, 2, \ldots)$.

It is well known that (1) has a solution if $\{f_i\}_{i\in\mathbf{I}}$ is a Riesz basis. Some of the problems discussed here have been solved in this special case by Zwaan [10]. Here we are interested in the more general case where $\{f_i\}_{i\in\mathbf{I}}$ is a frame. Since a frame is total, (1) has at most one solution in this case.

In all that follows, let $\{f_i\}_{i\in\mathbf{I}}$ be a frame with bounds A, B. As in Section 1, denote the frame operator by $S = TT^*$.

THEOREM 2.1. *Let $\{a_i\}_{i\in\mathbf{I}} \in l^2(\mathbf{I})$. There exists a unique element in $\mathcal{H}$ minimizing $\sum_{i\in\mathbf{I}} |a_i - \langle f, f_i\rangle|^2$; this element is $f = \sum_{i\in\mathbf{I}} a_i S^{-1} f_i$.*

Proof. Remember that the adjoint of T is given by T^*: $\mathcal{H} \to l^2(\mathbf{I}), T^* f = \{\langle f, f_i\rangle\}_{i\in\mathbf{I}}$. Now the theorem follows from [3, p. 59], where it is shown that the orthogonal projection of $\{a_i\}_{i\in\mathbf{I}}$ onto the range of T^* is given by

$$P\{a_i\}_{i\in\mathbf{I}} = \left\{ \left\langle \sum_{j\in\mathbf{I}} a_j S^{-1} f_j, f_i \right\rangle \right\}_{i\in\mathbf{I}}. \quad \blacksquare$$

In the sequel, let us call $\sum_{i\in\mathbf{I}} a_i S^{-1} f_i$ the b.a.s. (best approximation solution) of the moment problem (1). The explicit expression of the b.a.s. immediately shows what happens if the family $\{a_i\}_{i\in\mathbf{I}}$ is perturbed:

COROLLARY 2.2. *Let $\{a_i\}_{i\in\mathbf{I}}, \{a_i'\}_{i\in\mathbf{I}} \in l^2(\mathbf{I})$. Let f denote the b.a.s. of (1) and let f' be the b.a.s. of $\langle f, f_i\rangle = a_i'$. Then*

$$\|f - f'\|^2 \leqslant \frac{1}{A} \sum_{i\in\mathbf{I}} |a_i - a_i'|^2.$$

Next we consider the question of perturbation of the frame; what happens with the b.a.s. if a frame $\{g_i\}_{i\in\mathbf{I}}$ which is "close" to $\{f_i\}_{i\in\mathbf{I}}$ is substituted for $\{f_i\}_{i\in\mathbf{I}}$.

Corresponding to the frame $\{g_i\}_{i\in\mathbf{I}}$ we introduce the frame bounds A', B' and the frame operator $V = UU^*$. We consider U as a perturbation of T; with

$$K : l^2(\mathbf{I}) \to \mathcal{H}, \quad K\{c_i\} := \sum_{i\in\mathbf{I}} c_i(g_i - f_i)$$

we have

$$U = T + K.$$

LEMMA 2.3. (1) $\|S - V\| \leqslant (\sqrt{B} + \sqrt{B'})\|K\|$.
(2) $\|S^{-1} - V^{-1}\| \leqslant ((\sqrt{B} + \sqrt{B'})/AA')\|K\|$.

Proof. (1) $V - S = UU^* - TT^* = UU^* - UT^* + UT^* - TT^* = UK^* + KT^*$. So $\|V - S\| \leqslant (\|U\| + \|T\|)\|K\| \leqslant (\sqrt{B} + \sqrt{B'})\|K\|$.
(2) $\|S^{-1} - V^{-1}\| = \|S^{-1}(S - V)V^{-1}\| \leqslant \|S^{-1}\| \cdot \|S - V\| \cdot \|V^{-1}\| \leqslant ((\sqrt{B} + \sqrt{B'})/AA')\|K\|$. $\blacksquare$

Now, again let f be the b.a.s. of (1) and let f' be the b.a.s. of

$$\langle f, g_i\rangle = a_i, \quad i \in \mathbf{I}.$$

THEOREM 2.4. $\|f - f'\| \leqslant (1/A)\|K\{a_i\}_{i\in\mathbf{I}}\| + \|S^{-1} - V^{-1}\| \cdot \|U\{a_i\}_{i\in\mathbf{I}}\|$.

Proof. By Theorem 2.1,

$$f = \sum_{i\in\mathbf{I}} a_i S^{-1} f_i = S^{-1} T\{a_i\}_{i\in\mathbf{I}},$$

$$f' = \sum_{i\in\mathbf{I}} a_i V^{-1} g_i = V^{-1} U\{a_i\}_{i\in\mathbf{I}}.$$

So

$$\begin{aligned}
\|f - f'\| &= \|V^{-1}U\{a_i\}_{i\in\mathbf{I}} - S^{-1}T\{a_i\}_{i\in\mathbf{I}}\| \\
&\leqslant \|V^{-1}U\{a_i\}_{i\in\mathbf{I}} - S^{-1}U\{a_i\}_{i\in\mathbf{I}}\| \\
&\quad + \|S^{-1}U\{a_i\}_{i\in\mathbf{I}} - S^{-1}T\{a_i\}_{i\in\mathbf{I}}\| \\
&\leqslant \|V^{-1} - S^{-1}\| \cdot \|U\{a_i\}_{i\in\mathbf{I}}\| \\
&\quad + \|S^{-1}\| \cdot \|U\{a_i\}_{i\in\mathbf{I}} - T\{a_i\}_{i\in\mathbf{I}}\| \\
&\leqslant \|V^{-1} - S^{-1}\| \cdot \|U\{a_i\}_{i\in\mathbf{I}}\| + \frac{1}{A}\|K\{a_i\}_{i\in\mathbf{I}}\|. \quad \blacksquare
\end{aligned}$$

Using Lemma 2.3 we obtain an estimate involving $\|K\|$:

COROLLARY 2.5. $\|f - f'\| \leqslant [((\sqrt{B} + \sqrt{B'})/AA')\sqrt{B'} + 1/A] \cdot \|\{a_i\}_{i\in\mathbf{I}}\| \cdot \|K\|$. In the next sections we show how to estimate $\|K\|$ in the case of irregular sampling and Gabor frames. For the general case, observe that

$$\begin{aligned}
\|K\|^2 &= \|KK^*\| = \sup_{\|f\|=1} |\langle KK^* f, f\rangle| \\
&= \sup_{\|f\|=1} \sum_{i\in\mathbf{I}} |\langle f, g_i - f_i\rangle|^2.
\end{aligned}$$

The operator K also plays an important role in other contexts. For later use we mention a result from [1]; for convenience we use the index $\mathbf{I} = \mathbf{N}$, but a similar statement is true for any countable index set:

THEOREM 2.6. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame with bounds A, B. Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and suppose that*

$$\exists \lambda, \mu \geqslant 0 : \lambda + \frac{\mu}{\sqrt{A}} < 1 \quad \text{and}$$

$$\left\| \sum_{i=1}^{n} c_i (g_i - f_i) \right\| \leqslant \lambda \cdot \left\| \sum_{i=1}^{n} c_i f_i \right\| + \mu \cdot \left[\sum_{i=1}^{n} |c_i|^2 \right]^{1/2}$$

for all $c_1, \ldots, c_n (n = 1, 2, \ldots)$. Then $\{g_i\}_{i=1}^{\infty}$ is a frame with bounds $A(1 - (\lambda + \mu/\sqrt{A}))^2$ and $B(1 + \lambda + \mu/\sqrt{B})^2$.

In terms of the operators considered here the condition just means that

$$\|K\{c_i\}\| \leqslant \lambda \cdot \|T\{c_i\}\| + \mu \cdot \|\{c_i\}\|, \quad \forall \{c_i\} \in l^2(\mathbf{I}).$$

COROLLARY 2.7. *If $\|K\| < \sqrt{A}$, then $\{g_i\}_{i \in \mathbf{I}}$ is a frame with bounds $A(1 - \|K\|/\sqrt{A})^2, B(1 + \|K\|/\sqrt{B})^2$.*

Remarks. (1) Usually it is much more difficult to show that a family $\{g_i\}_{i \in \mathbf{I}}$ satisfies the lower frame condition than to show that $\{g_i\}_{i \in \mathbf{I}}$ is a Bessel sequence. Corollary 2.7 shows that the "difficult problem" reduces to the "easy problem" in the case of perturbations: if $\{f_i\}_{i \in \mathbf{I}}$ is a frame with lower bound A, then $\{g_i\}_{i \in \mathbf{I}}$ is a frame if $\{f_i - g_i\}_{i \in \mathbf{I}}$ is a Bessel sequence with bound smaller than A.

(2) Theorem 2.6 is much stronger than [2, Proposition 2.6]. Our applications in Section 3 and 4 could not be realized with the result from [2].

3. AN APPLICATION: IRREGULAR SAMPLING

For $f \in (L^1 \cap L^2)(\mathbf{R})$ we define the Fourier transformation by

$$\hat{f}(\xi) := \int_{\mathbf{R}} f(x) e^{-ix\xi} \, dx, \quad \xi \in \mathbf{R}.$$

As usual we extend the Fourier transformation to $L^2(\mathbf{R})$. Given $\omega > 0$ we consider the space of bandlimited functions

$$B^2(-\omega, \omega) := \{f \in L^2(\mathbf{R}) | \text{supp } \hat{f} \subseteq [-\omega, \omega]\}.$$

$B^2(-\omega, \omega)$ is a Hilbert space with respect to the inner product on $L^2(\mathbf{R})$.

We need a lemma [9, p. 181] :

LEMMA 3.1. *Let $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ be sequences of real numbers and suppose that there exist positive numbers B and γ such that*

$$\sum_{i \in \mathbf{N}} |f(x_i)|^2 \leqslant B \|f\|^2, \quad \forall f \in B^2(-\omega, \omega)$$

and

$$|x_i - y_i| \leqslant \gamma, \quad \forall i \in \mathbf{N}.$$

Then

$$\sum_{i \in \mathbf{N}} |f(x_i) - f(y_i)|^2 \leqslant B(e^{\omega\gamma} - 1)^2 \cdot \|f\|^2, \quad \forall f \in B^2(-\omega, \omega).$$

Let us now consider the sinc_ω function, $\text{sinc}_\omega(x) = \sin(\omega x)/\omega x$ for $x \neq 0$ and $\text{sinc}_\omega(0) = 1$. We are interested in frames consisting of translated versions of the sinc_ω function. Given $y \in \mathbf{R}$ we define the translation operator

$$L_y : B^2(-\omega, \omega) \to B^2(-\omega, \omega), \quad (Lyf)(x) := f(x-y), \ x \in \mathbf{R}.$$

It is well known that

$$\frac{\omega}{\pi} \langle f, L_y \text{sinc}_\omega \rangle = f(y), \quad \forall f \in B^2(-\omega, \omega), \forall y \in \mathbf{N}.$$

Given sampling sets $\{x_i\}_{i=1}^{\infty}, \{y_i\}_{i=1}^{\infty} \subseteq \mathbf{R}$ we define

$$\{f_i\}_{i=1}^{\infty} := \{L_{x_i} \text{sinc}_\omega\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} := \{L_{y_i} \text{sinc}_\omega\}_{i=1}^{\infty}.$$

Corresponding to $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ we use the notation from Section 2.

THEOREM 3.2. *Suppose that $\{f_i\}_{i=1}^{\infty}$ is a frame with bounds A, B and that there exists a positive constant γ such that $|x_i - y_i| \leqslant \gamma, \forall i \in \mathbf{N}$. Then*

(1) The operator K is well defined and bounded, $\|K\|^2 \leqslant B(e^{\gamma\omega} - 1)^2$.

(2) $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence,

$$\sum_{i \in \mathbf{N}} |\langle f, g_i \rangle|^2 \leqslant B e^{2\gamma\omega} \|f\|^2, \quad \forall f \in B^2(-\omega, \omega).$$

(3) If $\gamma < \ln(\sqrt{A/B} + 1)/\omega$, then $\{g_i\}_{i=1}^{\infty}$ is a frame with bounds $A(1 - \sqrt{B/A}(e^{\gamma\omega} - 1))^2, Be^{2\gamma\omega}$.

Proof. (1) By assumption,

$$\sum_{i \in \mathbf{N}} |\langle f, f_i \rangle|^2 = \frac{\pi^2}{\omega^2} \sum_{i \in \mathbf{N}} |f(x_i)|^2 \leqslant B \|f\|^2, \quad \forall f \in B^2(-\omega, \omega).$$

By Lemma 3.1,

$$\sum_{i \in \mathbf{N}} |\langle f, g_i \rangle - \langle f, f_i \rangle|^2 \leqslant B(e^{\gamma\omega} - 1)^2 \|f\|^2, \quad \forall f \in B^2(-\omega, \omega).$$

Using the fact that $|\langle f, g_i \rangle| \leqslant |\langle f, g_i - f_i \rangle| + |\langle f, f_i \rangle|$ it is clear that $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence. So the operator K is well defined and bounded on $l^2(\mathbf{N})$. We have

$$\|K\|^2 = \sup_{\|f\|=1} \sum_{i \in \mathbf{N}} |\langle f, f_i - g_i \rangle|^2 \leqslant B(e^{\gamma\omega} - 1)^2.$$

(2) Since $U = T + K$,

$$\|U\|^2 \leqslant (\|T\| + \|K\|)^2 \leqslant B(1 + e^{\gamma\omega} - 1)^2 = Be^{2\gamma\omega}.$$

That is

$$\sum_{i\in\mathbf{N}} |\langle f, g_i\rangle|^2 = \|U^* f\|^2 \leqslant Be^{2\gamma\omega}\|f\|^2, \quad \forall f \in B^2(-\omega,\omega).$$

(3) By Corollary 2.7 $\{g_i\}_{i=1}^{\infty}$ is a frame if $\|K\| < \sqrt{A}$; this is the case if $\sqrt{B}(e^{\gamma\omega} - 1) < \sqrt{A}$, that is, if $\gamma < \ln(\sqrt{A/B}+1)/\omega$. If the condition is satisfied we can use the frame bounds $A(1-\|K\|/\sqrt{A})^2 \geqslant A(1-\sqrt{B/A}(e^{\gamma\omega}-1))^2$ and $B(1+\|K\|/\sqrt{B})^2 \leqslant Be^{2\gamma\omega}$. ■

Remark. Theorem 3.2 is strongly related to the classical work of Duffin and Schaefer [6]. A similar statement could have been written down using [6, Lemma 2, 3] and the isomorphism

$$U\colon L^2(-\omega,\omega) \to B^2(-\omega,\omega), \quad Uf = \hat{f}.$$

However our approach gives the concrete bound $\gamma < \ln(\sqrt{\frac{A}{B}}+1)/\omega$, where Duffin and Schaefer [6] only obtain such a result for the case of regular sampling. Furthermore the use of Corollary 2.7 gives us the explicit frame bounds in Theorem 3.2(3).

Now let $\{f_i\}_{i=1}^{\infty} = \{L_{x_i}\mathrm{sinc}_\omega\}_{i=1}^{\infty}$ be a frame with bounds A, B and let $\{g_i\}_{i=1}^{\infty} = \{L_{y_i}\mathrm{sinc}_\omega\}_{i=1}^{\infty}$ be a perturbed frame such that

$$|x_i - y_i| \leqslant \gamma < \frac{\ln(\sqrt{A/B}+1)}{\omega}, \quad \forall i \in \mathbf{N}.$$

Then we have all the information needed to obtain an explicit estimate in Corollary 2.5 in terms of A, B, and γ :

COROLLARY 3.3.

$$\|f - f'\| \leqslant \left[\frac{B}{A}\frac{1+e^{\gamma\omega}}{(1-\sqrt{B/A}(e^{\gamma\omega}-1))^2}e^{\gamma\omega}+1\right] \times \frac{\sqrt{B}(e^{\gamma\omega}-1)}{A}\cdot\|\{a_i\}_{i\in\mathbf{N}}\|.$$

Proof. The estimate of $\|f - f'\|$ immediately gives an estimate of the supremum norm $\|f - f'\|_\infty$ since

$$\|f\|_\infty \leqslant \sqrt{2\omega}\cdot\|f\|, \quad \forall f \in B^2(-\omega,\omega).$$

The norm $\|f - f'\|$ is sometimes called the *time jitter error* [10]. Consider the variable x as the time. The moment problem $\langle f, f_i\rangle = a_i, \forall i \in \mathbf{N}$, is equivalent to

$$f(x_i) = a_i\frac{\omega}{\pi} \quad \forall i \in \mathbf{N},$$

so we are looking for a function f having the value $a_i(\omega/\pi)$ at time x_i. If x_i is perturbed to y_i, Theorem 2.1 give us the function f' having the value $a_i(\omega/\pi)$ at the time y_i (if such a function exists). Our result measures the distance between the "correct function" f and the "wrong function" f'. ■

We suggest the reader interested in irregular sampling look into [8]. The relation between [8] and the present paper is that a frame consisting of linear in dependent elements is a basis.

4. GABOR FRAMES

Let G be a group and let π be a representation of G on the Hilbert space $\mathcal{H}$. One may ask whether there exist a family $\{x_i\}_{i\in\mathbf{I}} \subseteq G$ and an $f \in \mathcal{H}$ such that $\{\pi(x_i)f\}_{i\in\mathbf{I}}$ is a frame for $\mathcal{H}$. A frame of this type is called a *coherent frame.* As an application of the theory from Section 2 we discuss two questions about perturbation of the *mother wavelet* f :

(1) Given a frame $\{\pi(x_i)f\}_{i\in\mathbf{I}}$ and $g \in L^2(R)$ which is "close" to f, is $\{\pi(x_i)g\}_{i\in\mathbf{I}}$ also a frame?

(2) If $\{\pi(x_i)f\}_{i\in\mathbf{I}}$ and $\{\pi(x_i)g\}_{i\in\mathbf{I}}$ are frames as in (1), how does the b.a.s. of a moment problem changes if $\{\pi(x_i)f\}_{i\in\mathbf{I}}$ is replaced by $\{\pi(x_i)g\}_{i\in\mathbf{I}}$?

We concentrate our interest on Gabor frames, but it will be clear that exactly the same technique can be applied to wavelet frames. Both have been studied intensively in the literature and there exist many results about conditions for a family to be a Gabor frame or a wavelet frame [4, 5, 7].

Let $\mathbf{G} = \mathbf{R}\times\mathbf{R}\times\Pi$ be the Heisenberg group and let π be the Schrödinger representation of $\mathbf{G}$ on $L^2(\mathbf{R})$:

$$[\pi(x,y,t)f](z) = te^{iy(z-x)}f(z-x), \quad f \in L^2(\mathbf{R}), z \in \mathbf{R}.$$

Given $f \in L^2(\mathbf{R})$ we define

$$f_{n,m} := \pi(na, 2\pi mb, 1)f, \quad n, m \in \mathbf{Z}.$$

We need a lemma, which is implicit in [7, Theorem 4.1.2] :

LEMMA 1. *Let $h \in L^2(\mathbf{R})$ and suppose that*
(1) Supp$(h) \subseteq I$, *where I is an interval of length $1/b$.*
(2) $\sum_{n\in\mathbf{Z}} |h(x-na)|^2 \leqslant \lambda, \forall x \in \mathbf{R}$.
Then $\{h_{n,m}\}_{(n,m)\in Z^2}$ is a Bessel sequence with upper bound λ/b.

It is not essential that the support of h be contained in an interval of length $1/b$.

LEMMA 4.2. *Let $h \in L^2(\mathbf{R})$ and suppose that*
(1) supp$(h) \subseteq I$, *where I is an interval of length k/b, where $k \in \mathbf{N}$.*
(2) $\sum_{n\in\mathbf{Z}} |h(x-na)|^2 \leqslant \lambda, \forall x \in \mathbf{R}$.
Then $\{h_{n,m}\}_{(n,m)\in Z^2}$ is a Bessel sequence with bound $k^2\lambda/b$.

Proof. Write $\mathbf{I}$ as a disjoint union of intervals of length $1/b$.

$$I = \bigcup_{i=1}^{k} I_i.$$

We define a family of functions and a corresponding family of operators:

$$h_j := h \cdot \mathbf{1}_{I_j}, \quad j = 1, 2, \ldots, k.$$

$$K_j : l^2(\mathbf{Z}^2) \to \mathscr{H}, K_j\{c_{n,m}\} = \sum_{(n,m)\in\mathbf{Z}^2} c_{n,m}(h_j)_{n,m}, \qquad j = 1, \ldots k.$$

By Lemma 4.1 $\|K_j\|^2 = \sup_{\|f\|=1} \sum_{(n,m)\in\mathbf{Z}^2} |\langle f, (h_j)_{n,m}\rangle|^2 \leqslant \lambda/b, j = 1, \ldots k$. Now, using that $h = \sum_{j=1}^{k} h_j$, it follows that

$$\left\| \sum_{(n,m)\in Z^2} c_{n,m} h_{n,m} \right\| \leqslant \sum_{j=1}^{k} \|K_j\| \cdot \|\{c_{n,m}\}\| \leqslant k\sqrt{\frac{\lambda}{b}} \cdot \|\{c_{n,m}\}\|.$$

■

THEOREM 1. *Suppose that* $\{f_{n,m}\}_{(n,m)\in\mathbf{Z}^2}$ *is a frame with bounds* A, B. *Let* $g \in L^2(\mathbf{R})$ *and suppose that*

(1) $\operatorname{supp}(f - g) \subseteq I$, *where* I *is an interval of length* $k/b, k \in \mathbf{N}$.

(2) $\sum_{n\in\mathbf{Z}} |(f-g)(x - na)|^2 \leqslant \lambda, \forall x \in \mathbf{R}$.

Then

(i) With $K\{c_{n,m}\} = \sum_{(n,m)\in\mathbf{Z}^2} c_{n,m}(f_{n,m} - g_{n,m})$,

$$\|K\|^2 = \sup_{\|h\|=1} \sum_{(n,m)\in\mathbf{Z}^2} |\langle h, (f-g)_{n,m}\rangle|^2 \leqslant \frac{k^2\lambda}{b}.$$

(ii) If $\lambda < Ab/k^2$, *then* $\{g_{n,m}\}_{(n,m)\in\mathbf{Z}^2}$ *is a frame with bounds* $A(1 - k\sqrt{\lambda/bA})^2, B(1 + k\sqrt{\lambda/bB})^2$.

The theorem follows from Lemma 4.2 and Corollary 2.7. If $\{f_{n,m}\}_{(n,m)\in\mathbf{Z}^2}$ is a frame and $g \in L^2(\mathbf{R})$ is chosen so that the conditions in Theorem 4.3 are satisfied we have all the information needed to obtain an explicit estimate in Corollary 2.5:

COROLLARY 4.4.

$$\|f - f'\| \leqslant \left[\frac{B}{A} \frac{(2 + k\sqrt{\lambda/bB})(1 + k\sqrt{\lambda/bB})}{(1 - k\sqrt{\lambda/bA})^2} + 1 \right] \times \frac{k}{A}\sqrt{\frac{\lambda}{b}} \|\{a_{n,m}\}_{(n,m)\in\mathbf{Z}^2}\|.$$

In a similar way our results can be applied to other results about Gabor frames and wavelet frames, e.g., [4, 5, 7].

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